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This paper undertakes, through use of a continuous elastic theoretical pile model, to predict the static bearing capacity of example piles from dynamic measurements taken while driving the piles. The field measurements referred to, made on both reduced-scale and full-scale piles, consist of the force and acceleration measured as functions of time at the top of the piles during driving. The prediction scheme employs a four-parameter model of elastic and rigid-body pile response to the measured hammer force input. When this scheme is employed to match analytically the time-varying pile velocity and displacement derived from the acceleration measurements, it then also yields an estimate of the pile ultimate static bearing capacity valid just after driving. This bearing capacity is verified by direct comparison with field static tests. For cases where a "set-up" time after initial driving has occurred, reuse of the method reveals the change in bearing capacity realized by the pile. For this, at least one blow of redriving is required. Finally, a simplified approximation to the given scheme is presented for engineering use, suggesting, among other things, that a routine dynamic test may be employed to determine pile static bearing capacity.

While many authors have employed the wave equation in relation to pile driving, the present study does not borrow directly from their methods. A broad search through the literature, (1), covers a number of these papers however. Important contributors in this area have been Smith (2,3), Forehand and Reese (4), and Samson, Hirsch, and Lowery (5,6,7).

Note.—Discussion open until August 1, 1969. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Soil Mechanics and Foundations Division, Proceedings of the American Society of Civil Engineers, Vol. 95, No. SM 12, March, 1969. Manuscript was submitted for review for possible publication on March 4, 1968.

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3Numerals in parentheses refer to corresponding items in Appendix VI.
THE THEORETICAL PILE MODEL

Consider the pile model of length L shown in Fig. 1, where X represents the rigid body motion of the pile, x the distance coordinate along the pile of a system of axes moving with the pile and \( \xi \) the elastic deflection of the point \( x \) away from its rest position. Let concentrated forces, functions of time, \( F_T(t) \) and \( F_B(t) \), act upon the top and bottom respectively of the pile, and let \( R_L(x, t) \) be the distributed resistive force of the soil along the length of the pile.

The top force \( F_T(t) = \) the hammer force transmitted to the pile. \( F_B(t) = \) a concentrated reactive force due to the soil resistance; \( R_L(x, t) = \) a distributed frictional force. In the model to be considered here, \( F_T(t) \) will be taken as the actual input force of the hammer as measured at the pile top for example, by a suitable force transducer. (See Refs. 8 and 9 and Appendix II.)

\[
Z(x, t) = A' + \frac{x + \xi}{L} \tag{1}
\]

The basic governing equation for the pile is the elastic wave equation. Details of this are considered in Appendix I. At the pile top \( (x = 0) \) the total dynamic displacement under the hammer blow is expressed by \( Z(0, t) \)

in which

\[
Z = X + \xi \tag{1}
\]

is the sum of rigid body \( X \) and elastic \( \xi \) contributions. The net result (solution of the wave equation) may be stated in the form

\[
Z(0, t) = \frac{\overline{F}_c(t)}{L} + \frac{2}{L} \sum_{n=1}^{\infty} \overline{F}_c(n, t) \tag{2}
\]

FIG. 1.—THEORETICAL MODEL OF PILE WITH DRIVING AND RESISTIVE FORCES

\( F_B(t) \) will be taken as zero in the present study, its effect (if any) being included in the distributed resistive force \( R_L(x, t) \), which will be discussed subsequently.

The basic governing equation for the pile is the elastic wave equation. Details of this are considered in Appendix I. At the pile top \( (x = 0) \) the total dynamic displacement under the hammer blow is expressed by \( Z(0, t) \)

in which\( Z = X + \xi \tag{1} \)

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\[
Z(0, t) = \frac{\overline{F}_c(t)}{L} + \frac{2}{L} \sum_{n=1}^{\infty} \overline{F}_c(n, t) \tag{2}
\]

FIG. 2.—DRIVING (HAMMER) FORCES ON TEST PILES

Comments on the form assumed for these laws are also made in Appendix I. Essentially, in the examples treated in the present work, the laws are taken as \( (a) \) a simple distributed Coulomb resistance along the sides and \( (b) \) \( F_B = 0 \), i.e. no point bearing.

In calculation, the Coulomb resistance was taken to be very small until the time \( t_1 \), corresponding closely to the peak of the measured velocity response curve; the full resistance value was employed thereafter. This full resistance value is called \( C_0 \) and the determination of a proper value for it is the prin-
PILE NO 1
SOIL CONDITION: SAND AND SILT
TYPE: STEEL PIPE
SECTION AREA: 783 IN²
DIAM: 12 INCHES
LENGTH: 74 FEET
ULT STATIC TEST BEARING CAPACITY: 214 KIPS

CALCULATION PARAMETERS
\( C = 1225 \text{ KIPS} \)
\( K_o = 75 \text{ KSF} \)
\( t = 0.090 \text{ SECS} \)

FIG. 3.—COMPARISON OF MEASURED AND PREDICTED DYNAMIC RESPONSE OF PILE 1

PILE NO 2
SOIL CONDITION: SILT
TYPE: STEEL PIPE
DIAM: 12 INCHES
SECTION AREA: 783 IN²
LENGTH: 74 FEET
ULT STATIC TEST BEARING CAPACITY: 252 KIPS

CALCULATION PARAMETERS
\( C = 250 \text{ KIPS} \)
\( K_o = 25 \text{ KSF} \)
\( t = 0.090 \text{ SECS} \)

FIG. 4.—COMPARISON OF MEASURED AND PREDICTED DYNAMIC RESPONSE OF PILE 2

PILE NO 3
SOIL CONDITION: SAND AND SILT
TYPE: STEEL PIPE
DIAM: 12 INCHES
SECTION AREA: 783 IN²
LENGTH: 74 FEET
ULT STATIC TEST BEARING CAPACITY: 214 KIPS

CALCULATION PARAMETERS
\( C = 1225 \text{ KIPS} \)
\( K_o = 75 \text{ KSF} \)
\( t = 0.090 \text{ SECS} \)

FIG. 5.—COMPARISON OF MEASURED AND PREDICTED DYNAMIC RESPONSE OF PILE 3

PILE NO 3a
SOIL CONDITION: SILT
TYPE: STEEL PIPE
DIAM: 12 INCHES
SECTION AREA: 783 IN²
LENGTH: 74 FEET
ULT STATIC TEST BEARING CAPACITY: 252 KIPS

CALCULATION PARAMETERS
\( C = 250 \text{ KIPS} \)
\( K_o = 25 \text{ KSF} \)
\( t = 0.090 \text{ SECS} \)

FIG. 6.—COMPARISON OF MEASURED AND PREDICTED DYNAMIC RESPONSE OF PILE 3a

FIG. 7.—VARIATION OF DISPLACEMENT RESPONSE FOR THREE ASSUMED BEARING CAPACITIES FOR PILE 3a
The principal objective of the present work, since \( C_0 \) constitutes the static bearing capacity of the pile.

**FIELD WORK**

In the course of other portions (see Ref. 8) of the project of which the present study was a central part, a number of steel pipe test piles, both full-scale and reduced-scale, were first driven and subsequently load tested statically.

During the driving, two kinds of dynamic information were measured: acceleration, and strain near the top of the pile. Details of the instrumentation are given in Ref. 8 and in Appendix II. Essentially, a piezoelectric accelerometer and foil resistance strain gages were mounted on the sides of the pile near its top. Their processed and recorded electrical output versus time during a hammer blow constitute the main input data of the present study.

Examples of the resulting hammer force versus time curves obtained by this method are shown for five example piles in Fig. 2. Figs. 3 through 7 contain the corresponding measured acceleration records. (Other data are also present in these figures, as will be discussed at a later point.)

Data on pile static testing are detailed in Ref. 8 and discussed briefly in Appendix III. At the present point, it suffices to note that static testing was carried out by two different loading methods, both of which gave essentially the same ultimate, or failure, load of the pile. These methods may be briefly identified by the terms "constant rate of penetration" (CRP) and "maintained load" (ML) test. Since it is only the ultimate load which is presently of interest, emphasis on particular aspects of static load testing is omitted. Detailed information on this is reported in Refs. 8 and 9. In general, the time interval between obtaining measured dynamic data of acceleration and hammer force at the end of driving and performance of a static load test was reduced to a minimum of only a few hours. For cases where a "set-up" time after initial driving has occurred, a static load test was performed first, and within two or three days after completion, the above stated dynamic data were obtained.

Soil conditions surrounding the test piles discussed in this paper are specified in Appendix IV. For the soil conditions, the dynamic measurements alluded to above. The method for doing this will now be outlined.

**ANALYSIS PROCEDURE**

Four parameters were chosen as variable for regulating the analytical pile model. These are (see Appendix I): \( C_0 \), the earth resistance along the sides of the pile; \( K_0 \), a soil shear resistance parameter; \( \xi \), the damping ratio of longitudinal elastic modes in the pile under the assumed resistance law; and \( t_0 \), nominally the time for pile-top velocity to reach its maximum. Of these parameters, \( C_0 \) proved by far the most important, the others being "trimming" parameters only. (Choosing \( K_0 = 0, \xi = 0.9, t_0 = 1/c \) for alternatively \( t_0 = \text{time to reach peak velocity} \) would effectively remove these trimming parameters from further consideration.) For example, almost any reasonable value of \( K_0 \) was usable. The damping ratio \( \xi \) was taken as large: \( \xi = 0.9 \) and was varied very little from this value in all calculations. The time \( t_0 \) was taken near the value \( 1/c \), though not exactly at this value, but in the case of a full-scale pile at a value close to the time of the maximum measured velocity response. The measured value for the speed of sound in steel which was used was \( c = 17200 \) fps. For better match of predicted and measured velocity response it was found convenient to vary \( t_0 \) slightly.

The computer-aided procedure then used was as follows: given the measured force input and the theory developed in Appendix I, the best possible match between the theoretical and measured pile displacement was sought using any reasonable values for the four parameters mentioned. The measured displacement was obtained through double integration of the measured acceleration. Often an intermediate aid in determining a good displacement match was to bring computed velocity peaks into agreement with experimental ones. When a "best" match was judged to be obtained, the corresponding value of

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**BEARING CAPACITY**

- **PILE NO. 11 REDUCED SCALE**
  - SOIL CONDITION: SAND
  - TYPE: STEEL PIPE
  - DIAM: 3 INCHES
  - SECTION AREA: 170 IN²
  - LENGTH: 12 FT
  - ULT STATIC TEST BEARING CAPACITY: 95 KIPS

---

**CALCULATION PARAMETERS**

- \( C_0 = 99 \text{ KIPS} \)
- \( K_0 = 5 \text{ KIPS} \)
- \( \xi = 0.95 \)
- \( t_0 = 0.0008 \text{ SECS} \)

**FIG. 8—COMPARISON OF MEASURED AND PREDICTED DYNAMIC RESPONSE OF REDUCED-SCALE PILE IV**
COMPARISON OF THEORY AND EXPERIMENT

Fig. 2 presents top force versus time as measured for all experimental cases in the present study. Fig. 3 shows results for Pile 1, a full-scale steel pile 74 ft long. Theoretical and experimental results bear close resemblance, and the predicted static bearing capacity is \( C_0 = 225 \) kips as opposed to 214 kips measured in a static load test. \( C_0 \) is thus high by 5%.

Fig. 4 shows results for Pile II, another full-scale steel pipe pile 74 ft long. Here the match of theoretical and experimental curves is less satisfactory but the static capacity prediction remains fair: \( C_0 = 290 \) kips as opposed to a measured ultimate static value of 252 kips, the prediction being high by 15%.

Fig. 5 shows results for Pile III, with theory being a good match as to shape in displacement, velocity, and acceleration. The static capacity predicted is \( C_0 = 220 \) kips as opposed to a measured ultimate value of 204 kips, the prediction being high by 8%.

Fig. 6 presents results for Pile IIIa, the same pile as III but re-driven for a second time test after a ground "set-up" period of two weeks. The dynamic portion of ultimate static capacity coincides with the actual static test result in this instance. This example typifies pile load capacity increase with time.

Because of the excellence of the dynamic prediction in this case it also becomes a good object for test of the sensitivity of the prediction method to the value of \( C_0 \) chosen. Fig. 7 illustrates this, where values of \( C_0 = 232, 242, \) and 252 kips have been tried, with corresponding theoretical displacements compared to measured test results. It is clear from this that close displacement matching sharply delineates the proper \( C_0 \) value under the present method.

The elastic and rigid body contributions to pile velocity may be separately calculated. The elastic result contains almost all of the "oscillatory" portion of the response, whereas the rigid body contribution is one of almost a straight-line decreasing velocity, suggesting the action of a rigid body under constant deceleration from constant soil resistance.

More can be said of the new information revealed by analyses of this type. This will be reserved to a later point, after a simplified dynamic pile calculation has been suggested.

Fig. 8 illustrates the dynamic prediction method for a case of a very short pile (12 ft). The match achieved is the worst of all cases treated as far as curve shape is concerned, though general trends are preserved. However, the predicted ultimate static bearing capacity is \( C_0 = 0.9 \) kips as opposed to a measured 9.5 kips, the prediction being high by 4%.

The calculated rigid body and elastic contributions for this case are presented in Fig. 9. Here, in this short pile, the elastic contribution is seen to be relatively much less important. As seen from this figure, the method under use clearly delineates the different roles of both elastic and rigid body motions in the action of piles.

SIMPLIFIED METHODS OF ANALYSIS

In Ref. 9 a rigid pile model was employed. Briefly, the analysis method used therein was as follows:

Let \( M \) be the mass of the pile and \( \ddot{X} \) its acceleration; then the force balance in this instance. This example typifies pile load capacity increase with time.

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Simplified Methods of Analysis

In Ref. 9 a rigid pile model was employed. Briefly, the analysis method used therein was as follows:

Let \( M \) be the mass of the pile and \( \ddot{X} \) its acceleration; then the force balance under hammer force \( F_T(t) \) requires that

\[
F_T(t) = R_L L = M\ddot{X} \tag{3}
\]
which \( R_L \) is the soil resistance per unit length. Under the assumption that soil resistance has the simple form
\[
R_L X = C_0 + \alpha X + \alpha^2 X^2 + \cdots
\] (4)
be value of \( C_0 \) may be determined experimentally if \( F(t) \) and \( X \) are known at instant when the velocity \( X = 0 \), at which instant \( R_L X = C_0 \).

Use of this scheme does in fact yield reasonable order-of-magnitude results for \( C_0 \) as indicated in Ref. 9.

However, various causes act to reduce the accuracy of this method, the chief one being the elastic action of the pile and the typicall "oscillatory" or "shaky" appearance of the usual accelerometer record, which must be read to obtain \( X \) at the point where its time integral \( X \) passes through zero. The particularly rapid local variations of \( X \) in the region of \( X = 0 \) render the ending of \( X \) unless quite uncertain of interpretation.

One very useful scheme for the correction of these inaccuracies is to plot \( R_L X \) from Eq. 3 for all times of interest to a given hammer blow (i.e. several tens of milliseconds). A typical result of this kind is shown in Fig. 10, when, beginning approximately at time \( t_r = L/c \) and ending approximately

\[
\begin{array}{c|c|c|c|c}
\text{Table I.--BEARING CAPACITY RESULTS IN KIPS} \\
\hline 
\text{Pile No.} & C_0 & R^2(t_r) & N^t \text{avg} & \text{Static Test} \\
\hline 
I & 225 & 225 & 205 & 214 \\
II & 290 & 291 & 290 & 252 \\
III & 220 & 222 & 222 & 204 \\
IVA & 242 & 245 & 204 & 242 \\
IV & 9.9 & 10.5 & 9.3 & 9.5 \\
\hline
\end{array}
\]

\( X \) passes through zero, or for any chosen shorter time interval, the no average of \( R_L X(t) \) is obtained. This average proves to be quite close to the measured or the dynamically calculated value of \( C_0 \), especially if taken over an interval in the vicinity of \( t_0 \), where \( X \) passes through zero. The time average can often be made quite well by eye by simply fitting a straight line through the plot of \( R_L X(t) \).

This scheme, in fact, little different in effect from another variant which may also be used: a best fit straight line is fitted through the descending portion of the velocity curves like those in Figs. 3 through 6, and the slope of the line is considered to be the rigid body deceleration; then using Eq. 3, in which the value of \( F_R(t) \) is read off at the time when the fitted straight-line passes through zero, yields a value for \( R_L X \).

That such approximate schemes may work quite well is attested to by the results portrayed in Table 1. The first column identifies the piles discussed previously under the dynamic analysis method. Column 2 gives \( C_0 \) as found by the matching method. Column 3 gives the resistance calculated from only the rigid body portion of the predicted dynamic response at the point where the velocity passes through zero. Column 4 shows results of simplified averaging in the neighborhood of \( t_0 \) based on Eq. 3. Column 5 presents actual static test results for ultimate load bearing capacity.

It may, from these results, be observed that the very practical graphical averaging methods suggested do in fact yield good results in the cases considered and render the more elaborate computer analysis unnecessary except as a research tool. This is a decided advantage, suggesting the possibility of rapid engineering estimation of pile static bearing capacity in the field through the use of instrumental dynamic tests.

CONCLUSIONS AND DISCUSSION

While earlier analytical methods have already been presented in the literature for analyzing pile driving "by the wave equation," the present method directly combines and compares theory with corresponding dynamic experiment. The method offers the possibility of dynamic in place of static load testing for piles, at least in certain types of soil.

In the present case, the measured hammer force is taken as input, while the field measured pile elastic and rigid body combined acceleration, plus velocity and displacement derived therefrom are considered as output. The analytical method postulates the same input and predicts the output. When the match between actual and predicted output is good, the prediction of static soil resistance also appears to be good. In this work a continuous pile model—rather than a set of discrete spring-masses—was used. While the continuous constant-section pile model has somewhat analogous advantages to those of a continuous constant-section beam model, this aspect of the present study is secondary to the main point. The use of any suitable analysis method for matching theoretical to experimental output will in principle be acceptable. What is considered novel in the present paper is the demonstration that, under suitable analysis, dynamic test information can yield the static bearing capacity. For example, it would appear that, in Refs. 2-7, use could have been made directly of hammer force (as portrayed in Fig. 2), instead of initial ram velocity as input. This might have the effect of increasing the accuracy of the calculated dynamic response as well as the ultimate bearing capacity.

The present studies were made in the main, on piles in silty soil (see Appendix IV). The favorable results obtained may be a direct result of this fact. An intentional effort was made, however, to predict results independently of detailed knowledge of the soil. The points of the study is that this may be possible for dynamic tests just as it is for static tests. To what extent this proves to be true for all types of soil is a subject for continuing study. It may well require more detailed regard to the definition of the assumed resistance law along the pile. It is hoped that light may be shed upon this by the experience of others and that constructive comment may be elicited on this point. One of the additional questions is that of the pile toe resistance. For point-bearing piles a form for this, \( F_R(t) \), must be assumed. The present study has succeeded to a reasonable extent with a very simple over-all resistance law. Continuing study may improve this law.

The attempt here to achieve simple dynamic estimates of bearing capacity through more easily applied criteria has also yielded encouraging results. In particular, the salient characteristics of the pile activity under the hammer
through studies of the type pictured in Fig. 9, the relative roles of elastic and rigid body activity are distinguished. For both long and short piles the velocity profile at pile top reveals first a period of strong elastic compression of the entire pile, followed by a rigid body deceleration as the compressed pile penetrates the ground; upon this deceleration profile are superimposed elastic oscillations of a "minor" nature probably originating in the region of cap block-hammer interaction. (They are not due to waves running up and down the pile). In the case of long length, the pile "compresses up" like a long spring under steady load, up to the point of maximum velocity; then the compressed pile descends into the soil more or less like a rigid body. In the case of the short pile, the rigid body action is the principal one, with only a relatively small elastic effect exhibiting oscillations about a mean much nearer to zero. These considerations explain why the rigid body model of pile activity (9) provided adequate for a period of time in explaining short pile activity but failed to explain long pile activity adequately. Both cases are reasonably handled, however, by the simplified averaging methods outlined above.

For the authors, the present study has resulted in familiarization with the rapidly passing response profiles actually occurring over a few tens of milliseconds after the hammer impact. These rapidly evolving profiles are now both believable and physically explainable.

Two particular facets of pile response that have proved interesting, and perhaps unexpected, are the following:

First, the onset of the velocity response is a curve resembling the form of the curve of \( F_T(t) \). That this is not merely coincidental and in fact that the two curves are related by a constant of proportionality \( c^2/E \) is easily demonstrable (8) when the response to \( F_T(t) \) of an infinitely long elastic pile with no resistance is calculated. (A convenient device for this demonstration is the use of the Laplace transform in solving the elastic wave equation of the pile.) Thus, the early part of the velocity curve represents "what is going on elastically before the soil resistance takes hold." Further, when the experimentally measured velocity and \( F_T(t) \) curves part company, the effect of soil resistance evidently has begun to be felt. This occurs usually at or near the maximum velocity peak, at a time \( t = L/c \).

Second, the pile wave sent down by the hammer blow dissipates, for the most part, directly into the soil on the downward passage, and little reflection of the wave occurs. It is under these considerations that the pile may be visualized as "compressed down" and to "act as a rigid body" for a time thereafter (Fig. 9).

The emphasis in the present paper has been on the prediction of dynamic response for the purpose of predicting static bearing capacity. Driving hammer forces, as available, have been assumed adequate to the job; hence no guidelines are established in this work for determining the needed magnitude of available driving force or energy. However, it is clear from results like those shown in Fig. 2, that good knowledge of developed driving forces can be obtained by the methods used and correlated with other pile information, such as displacement, to determine driving energy and required hammer characteristics.

The hammer forces shown in Fig. 2 are those for a particular hammer impact of each pile considered in this paper. A recorded hammer force was considered to be useful for analysis when there existed a net permanent pile settlement after impact. Full delineation of this minimum penetration amount is still open to further study.

The present paper lays no claim to solving the problem of determining static capacity by dynamic measurement under all conceivable circumstances. It does, however, present more than coincidental evidence that such a problem may be well posed and that continued effort in this direction can result in even excellent and rapid dynamic predictions of pile static bearing capacity. The full range of applicability of the method as well as its limitations are objects of continuing study by the writers.

ACKNOWLEDGMENTS

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APPENDIX 1.—THEORY OF THE ELASTIC PILE

The basic governing equation is the classical, one-dimensional elastic wave equation, written for the pile with a resistance term \( F_T/AE \) included

\[
\frac{\partial^2 v}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} = \frac{1}{c^2} \frac{\partial T}{\partial x} + \frac{F_T}{AE} (0 < x < L) \quad (t > 0)
\]

where \( c = \sqrt{E/\rho} \) is the wave velocity in the pile material, \( \rho \) is the pile material density, \( E \) is the modulus of elasticity of this material, and \( A \) is the pile cross-sectional area. (Pile dead weight is neglected in Eq. 5. It is a negligible fraction of the ultimate pile static capacity).

The boundary conditions are

\[
AE \frac{\partial v}{\partial x} = - F_T(t) \quad (x = 0) \quad (t > 0)
\]

\[
AE \frac{\partial v}{\partial x} = - F_T(t) \quad (x = L) \quad (t > 0)
\]

The initial conditions are quiescent

\[
X(0) + \{v(x, 0) = 0 \quad (0 \leq x \leq L) \quad (t = 0)
\]

Through defining

\[
Z(x, t) = X(t) + \{v(x, t)
\]

\[
X(0) + \{Z(x, 0) = 0 \quad (0 \leq x \leq L) \quad (t = 0)
\]

(9)
the governing equation becomes

$$\frac{\partial^2 Z}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 Z}{\partial t^2} = \frac{R_L(x, t)}{AE}$$  \hspace{1cm} (9)$$

Now let the soil resistance take the form

$$R_L(x, t) = R_0(x, t) + D \tilde{Z}(x, t)$$  \hspace{1cm} (10)$$

in which $D = \delta AE$  \hspace{1cm} (11)

for an appropriate "damping coefficient" $\delta$ to be defined later.

Taking the Fourier cosine transform of Eq. 9 yields

$$\tilde{Z}_c(v, t) + c^2 \frac{\partial^2 \tilde{Z}_c(v, t)}{\partial t^2} + \left( \frac{\pi x}{L} \right)^2 \tilde{Z}_c(v, t) = \frac{c^2}{AE} \{-(1)^{n+1} F_B(t) \}
+ F_T(t) - \tilde{R}_c(v, t) = H(v, t)$$  \hspace{1cm} (12)

in which the barred quantities together with the subscript $c$ represent the finite Fourier cosine transform

$$\tilde{Z}_c(v, t) = \int_0^L Z(x, t) \cos \frac{n \pi x}{L} \, dx$$  \hspace{1cm} (13)$$

It is remarked in passing that the cosine transform is employed because of its appropriateness to the present boundary conditions.

Solutions of Eq. 12 is given by

$$\tilde{Z}_c(v, t) = \tilde{X}_c(t) = \int_0^1 H(v, \tau) e^{-\gamma(\tau-t)} \sin \beta(t-\tau) \, d\tau$$  \hspace{1cm} (14)$$

for $n = 1, 2, 3, \ldots$

in which $\gamma = \frac{c^2 t}{L}$

$$\beta = \frac{n \pi c}{L} \left[ 1 - \left( \frac{m c}{2n \pi} \right)^2 \right]^{1/2}$$

and $\delta$ is to be defined as

$$\delta = \frac{2n \pi c}{c L}$$

with $\xi$ = ratio of damping to critical, so that each mode of the response has the same damping ratio $\xi$.

For the case $\gamma = 0$ (Eq. 12) yields the rigid body contribution

$$\tilde{Z}_c(0, t) = \tilde{X}_c(t) = \frac{c^2}{AE} \{ - F_B(t) + F_T(t) - \tilde{R}_c(0, t) \} = H(0, t)$$  \hspace{1cm} (15)$$

from which $\tilde{X}_c$ and $\tilde{X}_c$ are obtained by successive time integrations.

The net solution for $Z(x, t)$, the total displacement of a point on the pile, is given by

$$Z(x, t) = \frac{\tilde{X}_c(t)}{L} + \frac{2}{L} \sum_{n=1}^{-n} \frac{\tilde{Y}_c(v, t) \cos \frac{n \pi x}{L}}{L}$$  \hspace{1cm} (16)$$

in particular, at $x = 0$, the pile top

$$Z(0, t) = \frac{\tilde{X}_c(t)}{L} + \frac{2}{L} \sum_{n=1}^{-n} \frac{\tilde{Y}_c(v, t) \cos \frac{n \pi x}{L}}{L}$$  \hspace{1cm} (17)$$

The form and application of the soil resistance force law given by Eq. 10 needs further elucidation. In the present work the constant contribution $R_0(x, t)$ will be defined and applied as an "active" force as follows

$$R_0(x, t) = \begin{cases} 0 & \text{for } 0 \leq t \leq t_1, \\ \frac{C_0}{L} & \text{for } t_1 \leq t \leq \infty \end{cases}$$  \hspace{1cm} (18)$$

in which $C_0$ is constant for all $x$ and $t \geq t_1$. The time $t_1$ will be arbitrarily taken as $t_1 = \pi c$, the time for an elastic wave to travel from top to bottom of the pile; however the exact value chosen for $t_1$ can be considered a parameter of the solution as used here.

![Fig. 11. Application of Soil Resistance Law](image-url)

FIG. 11.—APPLICATION OF SOIL RESISTANCE LAW

While many definitions of $R_0(x, t)$ might be made, and further experimentation in this direction is very worthwhile, the above definition has proven adequate in examples studied to date. In other words, a total static soil resistance of amount $C_0$ is assumed to begin acting instantaneously once the time $t_1 = \pi c$ has been reached. We note that some law of resistance which defines the time of coming into action is necessary in the present context. This simple scheme has worked well in a variety of examples. Under the stated assumption the resistance $R_0(x, t)$ has the form indicated in Fig. 11.

One slight modification of the above scheme for resistance $R_0$ has also been employed; though it has not proven to be of primary importance it is worth mentioning. The resistance law $R_0(x, t)$ was alternately taken in the form

$$R_0(x, t) = \frac{C_0}{L} \int_0^1 K(x) \mathcal{P}\left(\frac{t-\tau}{c}\right) \, d\tau$$  \hspace{1cm} (19)$$
or $K(x)$ a coefficient depending upon soil shear resistance at the pile-soil
interface. For simplicity $K$ was taken as a constant $K = K_0$ throughout the pile
length. It may be pointed out in advance of the examples considered that this
resistance law, essentially an "elasto-plastic" law, for time preceding $t = t_i$, had
small effect upon the response, whatever the value of $K$ chosen, only
causing small changes in the early and less significant portion of the calcu-
lated pile displacement response. The form of the corresponding soil resis-
tance law as a function of time is suggested in Fig. 12. The values of $K$ used
in the latter modified resistance law were simply taken as those measured at
the top of the pile.

It should be remarked that certain series convergence problems connected
with the computer evaluation of Eq. 14 necessitated special attention. These
problems were resolved essentially (a) by working directly with displace-
ments in the form given by Eq. 17 from which the velocity and acceleration
were obtained using a finite difference approximation scheme, and (b) by em-

![Graph](image)

**FIG. 12.—ALTERNATE APPLICATION OF SOIL RESISTANCE LAW**

laying piecewise linear approximations to the given input functions $H(x,t)$, together with an adequate number of terms to the series (Eq. 17).

---

**APPENDIX II.—INSTRUMENTATION**

To measure acceleration a quartz high-frequency accelerometer (Kistler 38A) was bolted directly to the side of the steel pipe pile, with a single bolt about 2 or 3 pile diameters below the cap block position. Output wires from this led first to a charge amplifier, the signal from which then entered a power amplifier; this latter drove the light beam galvanometers of a Honey-
well Visicorder which provided a photographic record on paper of events ver-
sus time. Galvanometers used were nominally of a 3500 cycle type with a
resulting response rapid enough to follow the high frequency accelerometer
needs. Typically, a high paper speed (80 in. per sec) was used when record-
ing a pile hammer blow.

To measure hammer force a variety of procedures was tried, all centering
round the use of foil resistance strain gages in a bridge-carrier amplifier
configuration. At first strain gages were mounted directly on the outside pile
surface. These proved satisfactory but as a rule delicate to apply under field
conditions. Later a reliable force transducer bolted to the top of the pile, be-
low the cap block and cushion assembly as shown in Fig. 13, was developed.
This latter development allowed more nearly permanent strain gage instru-
mentation with the advantages of laboratory, as against field-installation, of
the gages themselves. The entire force transducer was bolted to a steel plate
which in turn was welded to the pile. The force transducer was statically cal-
ibrated periodically in the laboratory. Recording the hammer force during

![Diagram](image)

**FIG. 13.—ASSEMBLY OF PILE DRIVING APPARATUS USED FOR RECORDING DYNAMIC IMPACT RESPONSES**

driving and upon restriking the pile was achieved by conditioning the strain
signal with a carrier amplifier which in turn drove a galvanometer of the
above-mentioned recorder.

The entire system was carried to the field in a small truck. Occasionally,
portable a-c power was needed, but in general, conventional a-c power was
available on all sites visited.

A point to be noted is that, while strain (force) and acceleration were de-
sired for the top of the pile, it was necessary to make these measurements
somewhat below the actual top (2 diam or 3 diam) to assure diffusing out of
the hammer force uniformly into the pile body as well as to provide more safety for the accelerometer.

It was also found useful to employ a double set of both strain gages and accelerometers and to average results. In the case of the accelerometers this provided some information on the effect of rocking of the top end of the pile.

APPENDIX III.—PILE STATIC TESTS

The ML Test for static bearing capacity followed the Standard Ohio Highway Department load test procedure which is based on a variant of the "Engineering News Formula." In this procedure an initial load of 4/5 \( R \) is applied, in which the "yield load is \( R = \frac{2E_H}{s} + C_1 \); \( E_H \) = rated hammer energy (weight times drop height ft-lb); \( s \) = pile penetration in inches per blow; and \( C_1 \) = a coefficient lying between 0.10 and 0.30. Ultimate load is taken as six times \( R \). An additional increment of 1/5 \( R \) is applied after all measurable settlement has "ceased" (i.e. less than 0.01 in. in 20 min). Additional increments of 1/5 \( R \) are applied in similar fashion thereafter, the elapsed waiting time being increased one hr for each load increment. The ML tests were terminated, for this study, only after ultimate load had been reached.

Upon completion of the conventional ML Test, a CRP Test was performed. The penetration rate of the pile during this test is maintained at a predetermined rate, varying from 0.002 in. per min to 0.010 in. per min. Again, the

<table>
<thead>
<tr>
<th>Depth, in feet</th>
<th>Aggregate</th>
<th>Coarse sand</th>
<th>Fine sand</th>
<th>Silt</th>
<th>Clay</th>
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Fig. 14 illustrates the typical form of load versus settlement plot obtained with the two types of test. Details may be found in Ref. 8.
APPENDIX IV.—SOIL CHARACTERISTICS

All of the piles reported on in this paper were driven in silty soil. This soil type is undoubtedly an important factor in the success of the prediction.

TABLE 4.—SOIL NEAR PILES III and IIIa

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TABLE 5.—SOIL NEAR PILE IV

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scheme employed to date. It remains to be shown that equal success can be gained with other soil types.

Details of the soil borings relative to each pile are given in Tables 2 through 5. In these Tables the following column headings and conventions are

BEARING CAPACITY

followed: Figures given for solids are percent by weight of total solids. Aggregate is defined as solids greater than 2 mm in size (i.e. not passing through 2 mm mesh openings). Coarse sand ranges from 0.42 to 2 mm; fine sand ranges from 0.074 to 0.42 mm; silt lies between 0.005 and 0.074 mm; clay measures below 0.005 mm. Water content is given by percent weight of sample.

APPENDIX V.—REFERENCES


APPENDIX VI.—NOTATION

The following symbols are used in this paper:

\[ A = \text{cross-sectional area of pile}; \]
\[ c = \text{speed of sound in steel pile}; \]
\[ C_o = \text{predicted ultimate static soil resistance}; \]
\[ D_t = \text{damaging coefficients}; \]
\[ E = \text{elastic modulus of pile}; \]
\[ F_h = \text{rated hammer energy}; \]
\[ F_T, F_B (U) = \text{measured forces at top and bottom of pile}; \]
Experimental and theoretical study of the combined action of soil and structure is a prerequisite in safe and economical design of a buried structure. When calculating the loads involved, it should be borne in mind that the two components form a single integrated system, with the soil serving as the medium transmitting the external forces to the structure and protecting it by arching and restraint.

The spherical dome is an example of the ideal structure of this type, capable of supporting high loads through axial-membrane stresses alone; however, the danger of buckling is an adverse factor preventing full utilization of the cross section, especially in cases of thin elements or ones with a low modulus of elasticity.

Herein the writer describes buckling tests carried out on model domes buried in sand; results are compared with recent data obtained for unburied (*free*) domes under hydrostatic load.

### BUCKLING OF UNBURIED DOMES

The problem of buckling of spherical domes under uniform radial load, with small, linear, symmetric deformation, was solved by Zoelly (1915) and Schwerin (1922). The classical derivation (9) yields

\[
\rho_{th} = \frac{2}{\sqrt{3(1 - \nu^2)}} \left( \frac{t}{R} \right)^2
\]

Note.—Discussion open until August 1, 1969. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Soil Mechanics and Foundations Division, Proceedings of the American Society of Civil Engineers, Vol. 95, No. SM2, March, 1969. Manuscript was submitted for review for possible publication on April 25, 1968.

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*Numerals in parentheses refer to corresponding items in the Appendix I.—References.*